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Superstabilization of Bose systems: I. Thermodynamic study

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Abstract
In this paper, we present a general method of superstabilization corresponding to add the 'forward scattering' interaction to some non-superstable model. Then, we evaluate the free-energy density, the grand-canonical pressure and particle density for the new (superstable) Bose gas using those for the (non-superstable) model. To sum up, this first part should give us the thermodynamic basis necessary to explain, in a subsequent paper, the main interest of this method: the restoration of the strong equivalence of ensembles (canonical/grand-canonical) without destroying the 'fundamental' thermodynamic properties (such as Bose condensation phenomenona) issued from the first non-superstable system.

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1. Introduction

For any gas, the interaction potential between particles cannot be arbitrary: it must become sufficiently weak with increasing inter-particle distance to ensure the thermodynamic limit is met, and it becomes repulsive on short distances to prevent collapse of an infinite number of particles [1]. A *necessary* property to imply the existence of the corresponding grand partition function is the condition of *stability,* which has to be verified by the interaction potential. For Fermi gases or classical systems of particles, this condition is sufficient to ensure the existence of thermodynamic functions for any chemical potential, but it is not for Bose systems. For Bose gases, such a *sufficient* condition is given by the concept of *superstable* interaction, see [1]. In fact, this superstability property could be ensured for any Bose gas,

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in an *artificial* way, by adding the *'forward scattering'* interaction on the corresponding *non-superstable* interaction. Of course, that procedure restores the convergence of the grand-canonical functions in the thermodynamic limit, but also ensures the strong equivalence of canonical and grand-canonical ensembles. However, this concept of equivalence of ensembles is not the purpose of this paper. Before we go further, we need to recall some facts concerning the *homogeneous* Bose system under full interaction in order to fix the notation.

For the interacting homogeneous gas of n spinless bosons with mass m enclosed in a cubic box $\Lambda = \times_{\alpha=1}^{d} L \subset \mathbb{R}^d$, the Hamiltonian is defined by a self-adjoint (SA) extension

$$H_\Lambda^{(n)} = \left\{ \sum_{j=1}^{n} \left(-\frac{\hbar^2 \Delta_j}{2m} \right) + \frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \Phi(x_i - x_j) \right\}_{\text{s-a}} \tag{1.1}$$

on the *symmetrized* n-particle Hilbert spaces

$$\mathcal{H}_B^{(n)} \equiv (L^2(\Lambda^n))_{\text{symm}} \qquad \mathcal{H}_B^{(0)} = \mathbb{C} \tag{1.2}$$

appropriate for bosons [1, 2]. Here $\Phi(x) = \Phi(\|x\|)$ denotes a (real) *two-body* interaction potential and we consider *periodic boundary conditions*.

Let us consider a *first* kind of translation-invariant interaction potential $\varphi(x)$, which satisfies the following assumptions:

(A) $\varphi(x) \in L^1(\mathbb{R}^d)$;
(B) its (real) Fourier transformation

$$v(q) = \int_{\mathbb{R}^d} \mathrm{d}^d x \, \varphi(x) \, \mathrm{e}^{-\mathrm{i}qx} \qquad q \in \mathbb{R}^d \tag{1.3}$$

satisfies: $v(0) > 0$ and $0 \leqslant v(q) \leqslant v(0)$ for $q \in \mathbb{R}^d$.

Note that $\varphi(x) = \varphi(\|x\|)$ implies $v(q) = v(\|q\|)$. Then, by (1.1) with $\varphi(x) = \Phi(x)$, the Hamiltonian of the system acting on the boson Fock space $\mathcal{F}_\Lambda^B$ can be written in the second quantized form as

$$H_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2, q \in \Lambda^*} v(q) a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \tag{1.4}$$

considering that $\varepsilon_k = \hbar^2 k^2 / 2m$ represents the one-particle energy spectrum, and where the sums run over the set

$$\Lambda^* = \left\{ k \in \mathbb{R}^d : k_\alpha = \frac{2\pi n_\alpha}{L}, n_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, 2, \ldots, d \right\}$$

because of periodic boundary conditions. Here, $a_k^\# = \{a_k^* \text{ or } a_k\}$ are the usual boson creation/annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}kx}$, $k \in \Lambda^*$, $x \in \Lambda$, acting on the boson Fock space

$$\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)} \tag{1.5}$$

where $\mathcal{H}_B^{(n)}$ is the boson space (1.2). Under assumptions (A) and (B) on the interaction potential $\varphi(x)$, the full Hamiltonian H_Λ (1.4) is superstable [1].

Nowadays, for a large class of interaction potentials $\varphi(x)$, the Bose gas (1.4) in full interaction remains thermodynamically unsolved, so that even the standard canonical or grand-canonical thermodynamic functions (free-energy density or pressure) are not found explicitly.

A way to extract some thermodynamic properties from the original model (1.4) could be, either using a very particular two-body potential $\varphi(x)$ (see [3–8]), or truncating the full interaction of (1.4), see, for example, the analysis of the weakly imperfect Bose gas used to derive the microscopic theory of superfluidity [9–13].

Thus, an example is given by the mean-field (MF) Hamiltonian H_Λ^{MF}. It corresponds to a constant two-body potential in the box Λ:

$$\varphi^{\mathrm{MF}}(x) = \frac{1}{V}\varphi_0\chi_\Lambda(x) = \begin{Bmatrix} \varphi_0/V, & \text{for} \quad x \in \Lambda \\ 0, & \text{for} \quad x \notin \Lambda \end{Bmatrix}. \tag{1.6}$$

Its (real) Fourier transform (1.3) is equal to

$$v(q) = \varphi_0\delta_{q,0} = \begin{Bmatrix} \varphi_0, & \text{for} \quad q = 0 \\ 0, & \text{for} \quad q \neq 0 \end{Bmatrix}. \tag{1.7}$$

Hence, $\varphi_0 = v(0) > 0$ and the full Hamiltonian (1.4) becomes

$$H_\Lambda^{\mathrm{MF}} \equiv T_\Lambda + U_\Lambda^{\mathrm{MF}} \tag{1.8}$$

with

$$T_\Lambda \equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k \tag{1.9}$$

$$U_\Lambda^{\mathrm{MF}} \equiv \frac{v(0)}{2V} \sum_{k_1,k_2\in\Lambda^*} a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1}. \tag{1.10}$$

In fact, using some interaction potential $v(q)$ (1.3), the MF interaction (1.10) consists of cutting-off the terms with $q \neq 0$ in the full interaction of (1.4). The interaction (1.10) is also called the '*forward scattering*' interaction.

Considering the particle number operator,

$$N_\Lambda \equiv \sum_{k\in\Lambda^*} a_k^* a_k \tag{1.11}$$

note that

$$U_\Lambda^{\mathrm{MF}} = \frac{v(0)}{2V}\left(N_\Lambda^2 - N_\Lambda\right) \tag{1.12}$$

and the second term in (1.12) has the order $o(V)$. Then, the thermodynamic (bulk) properties of the MF model (1.8) *coincide* with those described by the Hamiltonian

$$H_\Lambda^{\mathrm{IBG}} \equiv T_\Lambda + \frac{v(0)}{2V}N_\Lambda^2 \tag{1.13}$$

known as the *imperfect Bose-gas* (IBG) [14]. Analysed exhaustively in [15–20], the IBG (1.13) (or the MF model (1.8)) is one among the first diagonal models extracted from the full Hamiltonian (1.4), see [14, 21–25]. Its main interest remains that the pathological aspect of the *perfect Bose gas* (PBG, see (1.9)), i.e. the non-existence of the grand-canonical pressure for positive chemical potential, is removed by the interaction $v(0)N_\Lambda^2/2V$ (or (1.10)), without destroying the conventional Bose–Einstein condensation for dimensions $d \geqslant 3$, and without creating a gap in the spectrum (for further discussions, see [13]).

Then, the interaction (1.10) or

$$\frac{\lambda}{V}N_\Lambda^2, \lambda = \frac{v(0)}{2} > 0 \tag{1.14}$$

in (1.13) stabilize the kinetic part T_Λ (1.9) for any chemical potential in the grand-canonical ensemble. However, adding (1.14) to a *non-superstable* Hamiltonian H_Λ^X (for example,

$H_\Lambda^X = T_\Lambda$ (1.9)) does not seem to change the intrinsic thermodynamic properties of the original Hamiltonian H_Λ^X. For example, by analogy with the PBG (1.9) and the IBG (1.13) [16, 17, 20], the phenomenon of Bose condensation seems to persist in the new superstabilized model

$$H_\Lambda^{SX} \equiv H_\Lambda^X + \frac{\lambda}{V} N_\Lambda^2.$$

Therefore, the purpose of this paper is to present a general method of superstabilization based on the addition of the interaction (1.14) (or the 'forward scattering' interaction (1.10)) to a *non-superstable* model X. Then, assuming *sufficient* conditions (such as the weak equivalence of ensembles) on the *non-superstable* Bose system X, we explicit in this paper the thermodynamic functions (free-energy density, grand-canonical pressure and particle density) of the new *superstabilized* model SX.

Hence, in section 2, we define the two Hamiltonians, *non-superstable* (model X) and *superstable* (model SX), with their basic thermodynamic functions, but also some sufficient conditions on the Bose system X. Then, in section 3, we derive the thermodynamic fundamental functions of the new superstable model SX using those of the non-superstable model X, i.e. we evaluate the thermodynamic limit of the free-energy density and the grand-canonical pressure for the *superstable* model SX. Section 4 corresponds to a detailed explanation of the thermodynamic behaviour of the corresponding grand-canonical SX particle density. Then section 5 gives some direct applications as the thermodynamic properties of the superstable model SX in the grand-canonical ensemble for a fixed particle density. We reserve section 6 for concluding remarks and discussions. Some technical statements are presented in appendix A.

2. Set-up of the problem

We consider a system X of bosons of mass m enclosed in a cubic box $\Lambda = \times_{\alpha=1}^{d} L \subset \mathbb{R}^d$ with a volume $V \equiv |\Lambda| = L^d$, defined by some *non-superstable* Hamiltonian

$$H_\Lambda^X \equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + U_\Lambda^X = T_\Lambda + U_\Lambda^X \tag{2.1}$$

with $\varepsilon_k = \hbar^2 k^2/2m \geqslant 0$ which defines the one-particle energy spectrum of free bosons in modes $k \in \Lambda^*$, see (1.9). Here U_Λ^X is a *stable* interaction [1], i.e. $\exists B \geqslant 0$ so that,

$$U_\Lambda^X \geqslant -BN_\Lambda \tag{2.2}$$

where N_Λ (1.11) is the particle number operator in the box Λ. We also consider

$$\left[H_\Lambda^X, N_\Lambda\right] = 0. \tag{2.3}$$

Equation (2.3) means that there is a conservation of the particle number in the box Λ for the Bose gas X (2.1). This condition (2.3) is useful here in the canonical ensemble.

Then, we denote H_Λ^{SX} the corresponding Hamiltonian defined by

$$H_\Lambda^{SX} \equiv H_\Lambda^X + \frac{\lambda}{V} N_\Lambda^2 \qquad \lambda > 0. \tag{2.4}$$

Here, the two Hamiltonians H_Λ^X (2.1) and H_Λ^{SX} (2.4) are defined on the boson Fock space $\mathcal{F}_\Lambda^B$ (1.5). By (2.2), note that the Hamiltonian H_Λ^{SX} (2.4) or the corresponding interaction

$$U_\Lambda^{SX} \equiv U_\Lambda^X + \frac{\lambda}{V} N_\Lambda^2 \qquad \lambda > 0$$

i.e.

$$H_\Lambda^{SX} = T_\Lambda + U_\Lambda^{SX}$$

is *superstable* [1]:

$$U_\Lambda^{SX} \geqslant -BN_\Lambda + \frac{\lambda}{V} N_\Lambda^2 \qquad \lambda > 0 \tag{2.5}$$

see (2.1) and (2.4). Also note that (2.3) and (2.4) imply the conservation of the particle number in the box Λ for the Bose gas SX (2.4):

$$\left[H_\Lambda^{SX}, N_\Lambda\right] = 0.$$

To fix the notation, the *fixed* inverse temperature is defined by β, and using the canonical ensemble, the particle density is denoted by ρ. In the grand-canonical ensemble, we choose two different parameters for the chemical potential:

- α as the chemical potential used for the (non-superstable) model X (2.1);
- μ as the chemical potential used for the (superstable) model SX (2.4).

In fact, this notation, α and μ for the chemical potential, allows us to highlight the thermodynamic links between the Bose systems X (2.1) and SX (2.4). Then, we define the 'standard' thermodynamic functions in the canonical and grand-canonical ensembles:

- by $f_\Lambda^X(\beta,\rho)$ and $f_\Lambda^{SX}(\beta,\rho)$, the free-energy densities associated with the Hamiltonians H_Λ^X (2.1) and H_Λ^{SX} (2.4), respectively, i.e.

$$\begin{aligned} f_\Lambda^X(\beta,\rho) &\equiv -\frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{H}_B^{(n)}} \left(\left\{\mathrm{e}^{-\beta H_\Lambda^X}\right\}^{(n)}\right) \\ f_\Lambda^{SX}(\beta,\rho) &\equiv -\frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{H}_B^{(n)}} \left(\left\{\mathrm{e}^{-\beta H_\Lambda^{SX}}\right\}^{(n)}\right) \end{aligned} \tag{2.6}$$

where $n = [V\rho]$ denotes the integer part of $V\rho$ ($\rho > 0$), whereas we define by

$$A^{(n)} \equiv A\lceil \mathcal{H}_B^{(n)} \tag{2.7}$$

the restriction of any operator A acting on the boson Fock space $\mathcal{F}_\Lambda^B$(1.5) to $\mathcal{H}_B^{(n)}$ (1.2);
- by $p_\Lambda^X(\beta,\alpha)$ and $p_\Lambda^{SX}(\beta,\mu)$, the grand-canonical pressures associated with H_Λ^X (2.1) and H_Λ^{SX} (2.4) respectively, i.e.

$$\begin{aligned} p_\Lambda^X(\beta,\alpha) &\equiv \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B} \left(\mathrm{e}^{-\beta\left(H_\Lambda^X - \alpha N_\Lambda\right)}\right) \\ p_\Lambda^{SX}(\beta,\mu) &\equiv \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B} \left(\mathrm{e}^{-\beta\left(H_\Lambda^{SX} - \mu N_\Lambda\right)}\right) \end{aligned} \tag{2.8}$$

- by $\rho_\Lambda^X(\beta,\alpha)$ and $\rho_\Lambda^{SX}(\beta,\mu)$, the grand-canonical particle densities associated with H_Λ^X (2.1) and H_Λ^{SX} (2.4) respectively, i.e.

$$\begin{aligned} \rho_\Lambda^X(\beta,\alpha) &\equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^X} (\beta,\alpha) = \partial_\alpha p_\Lambda^X(\beta,\alpha) \\ \rho_\Lambda^{SX}(\beta,\mu) &\equiv \left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^{SX}} (\beta,\mu) = \partial_\mu p_\Lambda^{SX}(\beta,\mu). \end{aligned} \tag{2.9}$$

Here

$$\langle - \rangle_{H_\Lambda^X}(\beta,\alpha) \equiv \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left((-)\,\mathrm{e}^{-\beta(H_\Lambda^X-\alpha N_\Lambda)}\right)}{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^X-\alpha N_\Lambda)}\right)}$$
$$\langle - \rangle_{H_\Lambda^{SX}}(\beta,\mu) \equiv \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left((-)\,\mathrm{e}^{-\beta(H_\Lambda^{SX}-\mu N_\Lambda)}\right)}{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{SX}-\mu N_\Lambda)}\right)} \tag{2.10}$$

see (2.7), i.e. $\langle - \rangle_{H_\Lambda^X}(\beta,\alpha)$ or $\left(\langle - \rangle_{H_\Lambda^{SX}}(\beta,\mu)\right)$ represents the (finite-volume) *grand-canonical* Gibbs state for some Hamiltonian H_Λ^X (or H_Λ^{SX}).

Now, let us consider three assumptions that have to be verified by the thermodynamic limit of functions (2.6) and (2.8) for the first (non-superstable) Hamiltonian H_Λ^X (2.1).

Condition 2.1.

(i) The (infinite-volume) free-energy density

$$f^X(\beta,\rho) \equiv \lim_\Lambda f_\Lambda^X(\beta,\rho) < +\infty \tag{2.11}$$

cf (2.6), is defined for any $\beta > 0$ and $\rho > 0$.

(ii) The stability domain Q^X of H_Λ^X defined by

$$Q^X \equiv \{(\beta > 0, \alpha \in \mathbb{R}) : \lim_\Lambda p_\Lambda^X(\beta,\alpha) < +\infty\} \tag{2.12}$$

cf (2.8), is equal to

$$Q^X = Q \equiv \{\beta > 0\} \times \{\alpha < \alpha_{\mathrm{sup}} < +\infty\}. \tag{2.13}$$

(iii) Fixing the inverse temperature $\beta > 0$, the thermodynamic limit of $p_\Lambda^X(\beta,\alpha)$ (2.8), i.e.

$$p^X(\beta,\alpha) \equiv \lim_\Lambda p_\Lambda^X(\beta,\alpha) < +\infty \qquad \text{for} \quad \alpha < \alpha_{\mathrm{sup}} \tag{2.14}$$

and the (infinite-volume) free-energy density $f^X(\beta,\rho)$, (2.11), are always related by the Legendre transformation:

$$p^X(\beta,\alpha) = \sup_{\rho>0}\{\alpha\rho - f^X(\beta,\rho)\} \qquad \alpha < \alpha_{\mathrm{sup}}$$
$$f^X(\beta,\rho) = \sup_{\alpha<\alpha_{\mathrm{sup}}}\{\alpha\rho - p^X(\beta,\alpha)\} \qquad \rho > 0 \tag{2.15}$$

i.e. the weak equivalence of canonical and grand-canonical ensembles is verified for the gas X (2.1).

Of course, the PBG (1.9) verifies the assumptions of condition 2.1 with $\alpha_{\mathrm{sup}} = 0$, see [26].

We conclude these definitions and assumptions with a few remarks. The first one concerns condition (ii) and the existence of a chemical potential limit $\alpha_{\mathrm{sup}} \in \mathbb{R}$ in the stability domain Q^X (2.12). The stability domain Q^X may also be

$$Q^X = Q \cup \{(\beta > 0, \alpha_{\mathrm{sup}})\} = \{\beta > 0\} \times \{\alpha \leqslant \alpha_{\mathrm{sup}} < +\infty\} \tag{2.16}$$

instead of (2.13). A simple example is given in [27]. The case (2.16) implies that no problems are more complex than (2.13). To simplify, we consider by default that *only* (2.13) is verified.

Also, note that the corresponding PBG pressure in the thermodynamic limit is defined for $\alpha < \alpha_{\mathrm{sup}} = 0$, i.e. the stability domain Q^{PBG} of the PBG equals

$$Q^{\mathrm{PBG}} = \{\beta > 0\} \times \{\alpha < \alpha_{\mathrm{sup}} = 0\}$$

for $d < 3$, whereas for $d \geqslant 3$ this domain could be extended by continuity of the infinite-volume PBG pressure to

$$Q^{\mathrm{PBG}} \cup \{(\beta > 0, 0)\} = \{\beta > 0\} \times \{\alpha \leqslant 0\}$$

see below (5.18). Via (2.13), the infinite-volume pressure $p^X(\beta, \alpha)$ (2.14) is then well defined, only for $\alpha < \alpha_{\mathrm{sup}}$, but if

$$\lim_{\alpha \to \alpha_{\mathrm{sup}}^-} p^X(\beta, \alpha) < +\infty$$

then we extend $p^X(\beta, \alpha)$ by continuity to

$$Q^X \cup \{(\beta > 0, \alpha_{\mathrm{sup}})\} = Q \cup \{(\beta > 0, \alpha_{\mathrm{sup}})\} = \{\beta > 0\} \times \{\alpha \leqslant \alpha_{\mathrm{sup}} < +\infty\}$$

i.e.

$$p^X(\beta, \alpha_{\mathrm{sup}}) \equiv \lim_{\alpha \to \alpha_{\mathrm{sup}}^-} p^X(\beta, \alpha) < +\infty. \tag{2.17}$$

Note that $\left\{p_\Lambda^X(\beta, \alpha)\right\}_\Lambda$ is a set of convex functions for $\alpha < \alpha_{\mathrm{sup}}$, and using the Griffiths lemma [28, 29], by (2.9), the (infinite-volume) particle density $\rho^X(\beta, \alpha)$

$$\rho^X(\beta, \alpha) \equiv \lim_\Lambda \rho_\Lambda^X(\beta, \alpha) < +\infty \qquad \text{for} \quad \alpha < \alpha_{\mathrm{sup}} \tag{2.18}$$

equals

$$\rho^X(\beta, \alpha) = \partial_\alpha p^X(\beta, \alpha). \tag{2.19}$$

Then, we may have two different cases:

(a) either there is no critical particle density, i.e.

$$\lim_{\alpha \to \alpha_{\mathrm{sup}}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{\mathrm{sup}}^-} \partial_\alpha p^X(\beta, \alpha) = +\infty \tag{2.20}$$

cf (2.19),

(b) or there is a *saturation* of the infinite-volume particle density $\rho^X(\beta, \alpha)$ (2.18), i.e. one has a critical particle density

$$\rho^X(\beta, \alpha_{\mathrm{sup}}) \equiv \lim_{\alpha \to \alpha_{\mathrm{sup}}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{\mathrm{sup}}^-} \partial_\alpha p^X(\beta, \alpha) < +\infty \tag{2.21}$$

cf (2.19).

Our second remark concerns condition (iii). In fact, this property is verified by a large set of Hamiltonians. For example, in (1.1), we can consider a *second* kind of translation-invariant interaction potential $\widetilde{\varphi}(x) = \widetilde{\varphi}(r = \|x\|) = \Phi(x)$, in a three-dimensional space ($d = 3$), which verifies:

(A′) a sufficiently rapid decrease at infinity; i.e. for large $r > 0, \widetilde{\varphi}(r)| < 1/r^{3+\xi}$, with $\xi > 0$;
(B′) a sufficiently rapid increase at zero: $\widetilde{\varphi}(r) > 1/r^{3+\xi}$ for $r < r_0$ and $\xi > 0$;
(C′) bounded from below: $\widetilde{\varphi}(r) > A$.

Then, when these conditions are satisfied, for any distribution of points $(x_1, \ldots, x_n)$ in $\mathbb{R}^3$ we have

$$\frac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} \widetilde{\varphi}(x_i - x_j) > -Bn$$

where $B > 0$ is a constant only depending on $\widetilde{\varphi}(x)$ [30, 31]. Therefore, using the basic boson operators $a^{\#}(x) = \{a^*(x) \text{ or } a(x)\}$ on $\mathcal{F}_\Lambda^B$ (1.5), and assuming that (A'), (B') and (C') are verified, the Hamiltonian

$$\widetilde{H}_\Lambda \equiv \int_\Lambda a^*(x)\left(-\frac{\hbar^2\Delta}{2m}\right)a(x)\,\mathrm{d}^3x + \frac{1}{2}\int_\Lambda\int_\Lambda a^*(x)a^*(y)\widetilde{\varphi}(x-y)a(x)a(y)\,\mathrm{d}^3x\,\mathrm{d}^3y \tag{2.22}$$

corresponding in $\mathcal{H}_B^{(n)}$ (1.2) to $H_\Lambda^{(n)}$ (1.1) with $\Phi(x) = \widetilde{\varphi}(x)$, is *stable*, i.e. one has (2.2) for

$$U_\Lambda^X = \widetilde{U}_\Lambda \equiv \frac{1}{2}\int_\Lambda\int_\Lambda a^*(x)a^*(y)\widetilde{\varphi}(x-y)a(x)a(y)\,\mathrm{d}^3x\,\mathrm{d}^3y.$$

Actually, (i) is verified. Moreover, using the results about the related entropy done in [32], the condition (iii), i.e. the *weak equivalence* (2.15) of canonical and grand-canonical ensembles, is verified.

Now, assuming the necessary assumptions of condition 2.1, the objective of the final part of this paper is to compute the standard thermodynamic functions for the superstable Bose gas SX (2.4), using those for the non-superstable model X (2.1), see (2.6)–(2.9).

3. Free-energy density and grand-canonical pressure

3.1. Canonical ensemble: free-energy density

First, we evaluate, in the thermodynamic limit, the free-energy density $f_\Lambda^{SX}(\beta,\rho)$ from the thermodynamic limit of the free-energy density $f_\Lambda^X(\beta,\rho)$, cf (2.6).

Theorem 3.1. *Assuming (i) of condition 2.1, we have*

$$f^{SX}(\beta,\rho) = f^X(\beta,\rho) + \lambda\rho^2 \qquad \rho > 0 \tag{3.1}$$

with $f^X(\beta,\rho)$ defined by (2.11) and

$$f^{SX}(\beta,\rho) \equiv \lim_\Lambda f_\Lambda^{SX}(\beta,\rho) < +\infty \tag{3.2}$$

see (2.6).

Proof. Assumption (i) of condition 2.1 implies the existence of the thermodynamic limit $f^X(\beta,\rho)$ (2.11) for $\rho > 0$. Then, from (2.3), (2.4) and (2.6) one has

$$f_\Lambda^{SX}(\beta,\rho) = f_\Lambda^X(\beta,\rho) + \lambda\rho^2$$

which implies (3.1) in the thermodynamic limit. □

Note that if $f^X(\beta,\rho)$ is convex for $\rho > 0$ (condition 2.1 (iii)) then by (3.1), $f^{SX}(\beta,\rho)$ (3.2) is *strictly* convex for $\rho > 0$. Thus, the grand-canonical particle density $\rho_\Lambda^{SX}(\beta,\mu)$ (2.9) should be *continuous* in the thermodynamic limit as a function of μ.

3.2. The grand-canonical pressure

Now we establish the thermodynamic limit of the grand-canonical pressure $p_\Lambda^{SX}(\beta,\mu)$ from $p_\Lambda^X(\beta,\alpha)$, see (2.8).

Theorem 3.2. *If the non-superstable Hamiltonian H_Λ^X (2.1) verifies condition 2.1 then the stability domain Q^{SX} of H_Λ^{SX} (2.4) equals*

$$Q^{SX} \equiv \{(\beta > 0, \mu \in \mathbb{R}) : \lim_\Lambda p_\Lambda^{SX}(\beta,\mu) < +\infty\} = Q^S \equiv \{\beta > 0\} \times \{\mu \in \mathbb{R}\} \tag{3.3}$$

and

$$p^{SX}(\beta,\mu) \equiv \lim_{\Lambda} p^{SX}_{\Lambda}(\beta,\mu) = \inf_{\alpha<\alpha_{\sup}} \left\{ p^{X}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right\} \tag{3.4}$$

for $(\beta,\mu) \in Q^S$. *Here* $p^X(\beta,\alpha)$ *is defined by (2.14).*

Proof. From (2.2), the Hamiltonian (2.4) is superstable [1], see (2.5). So, the thermodynamic limit

$$p^{SX}(\beta,\mu) \equiv \lim_{\Lambda} p^{SX}_{\Lambda}(\beta,\mu)$$

exists for any $\beta > 0$ and $\mu \in \mathbb{R}$, i.e. one has (3.3).

The rest of the proof is a generalization for some Bose systems satisfying condition 2.1 of for a specific diagonal model, see [33]. In fact, by (2.15) in condition 2.1, the (infinite-volume) free-energy density $f^X(\beta,\rho)$ (2.11), as a function of $\rho > 0$, is always convex. Therefore, since $p^{SX}(\beta,\mu)$ is a continuous function for $\mu \in \mathbb{R}$, the lemmas A.2, A.4 and A.5 (appendix A) imply (3.4) for any $\rho > 0$. □

As a matter of fact, the main idea in proving theorem 3.2 is to consider the pressure $p^{SX}(\beta,\mu)$ as the Legendre transformation of $f^{SX}(\beta,\rho)$, (3.1) and (3.2), which, by direct computations, implies

$$p^{SX}(\beta,\mu) = \sup_{\rho>0} \{\mu\rho - \lambda\rho^2 - f^X(\beta,\rho)\} = \sup_{\rho>0} \left\{ \inf_{\alpha<\alpha_{\sup}} \left\{ \alpha\rho + \frac{(\mu-\alpha)^2}{4\lambda} - f^X(\beta,\rho) \right\} \right\}. \tag{3.5}$$

Then, the technical difficulty of the proof is to find some sufficient conditions, i.e. condition 2.1, to commute $\sup_{\rho>0}$ and $\inf_{\alpha<\alpha_{\sup}}$ in (3.5) to obtain

$$\begin{aligned} p^{SX}(\beta,\mu) &= \inf_{\alpha<\alpha_{\sup}} \left\{ \sup_{\rho>0} \left\{ \alpha\rho + \frac{(\mu-\alpha)^2}{4\lambda} - f^X(\beta,\rho) \right\} \right\} \\ &= \inf_{\alpha<\alpha_{\sup}} \left\{ \sup_{\rho>0} \{\alpha\rho - f^X(\beta,\rho)\} + \frac{(\mu-\alpha)^2}{4\lambda} \right\} \end{aligned} \tag{3.6}$$

i.e. (3.4) (see (2.15)).

4. Behaviour of the grand-canonical particle density

For most parts of the studied Hamiltonians, the corresponding particle density $\rho^X_\Lambda(\beta,\alpha)$ (2.9) is a continuous function for any $\alpha < \alpha_{\sup}$, even in the thermodynamic limit. Nevertheless, the *Bogoliubov weakly imperfect Bose gas* (see equation (3.81) in [10]) gives us an example for which there might exist one chemical potential $\alpha_{1,\beta} < \alpha_{\sup} = 0$, so that the corresponding grand-canonical particle density is *not* continuous at this point $\alpha_{1,\beta}$ in the thermodynamic limit [11–13]. To illustrate our purpose, we assume that the non-superstable Hamiltonian H^X_Λ (2.1) verifies the three assumptions (i)–(iii) of condition 2.1 and also the following additional condition:

(iv) the (infinite-volume) particle density $\rho^X(\beta,\alpha)$ (2.18) is a continuous function for $\alpha < \alpha_{\sup}$, except for one chemical potential $\alpha_{1,\beta} < \alpha_{\sup}$.

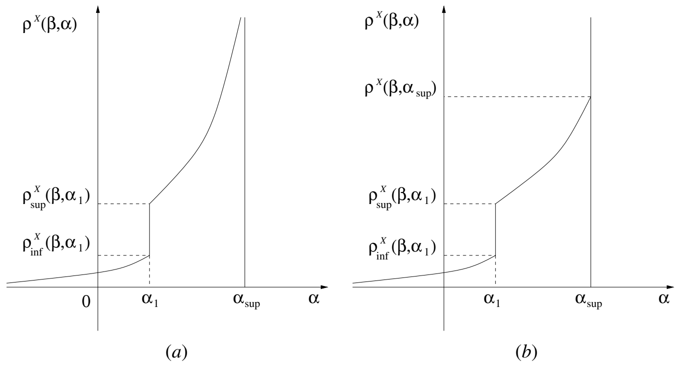

Figure 1. Illustration of the (infinite-volume) particle density $\rho^X(\beta, \alpha)$ (2.18) with (*a*) no saturation of the particle density $\rho^X(\beta, \alpha)$, see (2.20); (*b*) saturation of the particle density $\rho^X(\beta, \alpha)$, see (2.21).

Before we go further, let me add the following remark. Since the derivative (2.19) (as a function of $\alpha \neq \alpha_{1,\beta}$) of the pressure $p^X(\beta, \alpha \neq \alpha_{1,\beta})$ is continuous (see assumption (iv)), using a Tauberien theorem proven in [32], the existence of $p^X(\beta, \alpha)$ already implies the strict convexity of $f^X(\beta, \rho)$ (2.11) and the *weak equivalence* (2.15) of ensembles (canonical/grand-canonical) for

$$\rho \in I_\rho(\beta) \equiv \left(0, \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha)\right) \Big\backslash \left[\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right] \tag{4.1}$$

with

$$\begin{aligned} \rho^X_{\inf}(\beta, \alpha_{1,\beta}) &\equiv \lim_{\alpha \to \alpha_{1,\beta}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{1,\beta}^-} \partial_\alpha p^X(\beta, \alpha) < \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha) \\ \rho^X_{\sup}(\beta, \alpha_{1,\beta}) &\equiv \lim_{\alpha \to \alpha_{1,\beta}^+} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{1,\beta}^+} \partial_\alpha p^X(\beta, \alpha) < \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha). \end{aligned} \tag{4.2}$$

Therefore, if (ii) and (iv) are verified, conditions (i) and (iii) ensure the existence of the free-energy density $f^X(\beta, \rho)$ (2.11) and the weak equivalence (2.15) only for

$$\rho \in \left[\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right] \tag{4.3}$$

if there is a discontinuity of the particle density $\rho^X(\beta, \alpha)$ for some $\alpha = \alpha_{1,\beta}$, see (4.2), but also for

$$\rho \geqslant \rho^X(\beta, \alpha_{\sup}) \tag{4.4}$$

if there is a *saturation* of the infinite-volume particle density $\rho^X(\beta, \alpha)$, see (2.21). An illustration of $\rho^X(\beta, \alpha)$ is given in figure 1.

4.1. *Preliminary study*

For $\mu \in \mathbb{R}$, we analyse now the function $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$, solution of equation (3.4):

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\sup}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = p^X(\beta, \widetilde{\alpha}_\beta(\mu)) + \frac{(\mu - \widetilde{\alpha}_\beta(\mu))^2}{4\lambda}. \tag{4.5}$$

Since $p^X(\beta, \alpha)$ (2.14) is a convex function for $\alpha < \alpha_{\sup}$ [1, 34], the function

$$g(\alpha) \equiv \partial_\alpha p^X(\beta, \alpha) + \frac{\alpha}{2\lambda} \tag{4.6}$$

is a strictly increasing function for $\alpha < \alpha_{\mathrm{sup}}$. Note that $g(\alpha)$ is not continuous for $\alpha = \alpha_{1,\beta}$. Then, we define the two chemical potentials $\mu_{1,\mathrm{inf}}(\beta)$ and $\mu_{1,\mathrm{sup}}(\beta)$ by

$$\left\{\begin{array}{l} \mu_{1,\mathrm{inf}}(\beta) \equiv 2\lambda\rho^X_{\mathrm{inf}}(\beta,\alpha_{1,\beta}) + \alpha_{1,\beta} = 2\lambda \lim\limits_{\alpha\to\alpha^-_{1,\beta}} \partial_\alpha p^X(\beta,\alpha) + \alpha_{1,\beta} = 2\lambda \lim\limits_{\alpha\to\alpha^-_{1,\beta}} g(\alpha) \\ \mu_{1,\mathrm{sup}}(\beta) \equiv 2\lambda\rho^X_{\mathrm{sup}}(\beta,\alpha_{1,\beta}) + \alpha_{1,\beta} = 2\lambda \lim\limits_{\alpha\to\alpha^+_{1,\beta}} \partial_\alpha p^X(\beta,\alpha) + \alpha_{1,\beta} = 2\lambda \lim\limits_{\alpha\to\alpha^+_{1,\beta}} g(\alpha) \end{array}\right\} \tag{4.7}$$

cf (2.19) and (4.6), with $\rho^X_{\mathrm{inf}}(\beta,\alpha_{1,\beta})$ and $\rho^X_{\mathrm{sup}}(\beta,\alpha_{1,\beta})$ defined by (4.2) (see also figure 1). Then, for $\mu < \mu_{1,\mathrm{inf}}(\beta)$ or for μ such that

$$\mu_{1,\mathrm{sup}}(\beta) < \mu < 2\lambda \lim_{\alpha\to\alpha^-_{\mathrm{sup}}} \partial_\alpha p^X(\beta,\alpha) + \alpha_{\mathrm{sup}} \tag{4.8}$$

where $\widetilde{\alpha}_\beta(\mu)$ is defined by (4.5) is always the *unique* solution of

$$\left\{\partial_\alpha\left(p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right)\right\}\bigg|_{\alpha=\widetilde{\alpha}_\beta(\mu)} = 0 \tag{4.9}$$

i.e. by (4.6) $\widetilde{\alpha}_\beta(\mu)$ verifies

$$g(\widetilde{\alpha}_\beta(\mu)) \equiv \partial_\alpha p^X(\beta,\widetilde{\alpha}_\beta(\mu)) + \frac{\widetilde{\alpha}_\beta(\mu)}{2\lambda} = \frac{\mu}{2\lambda}.$$

However, for $\mu \in [\mu_{1,\mathrm{inf}}(\beta), \mu_{1,\mathrm{sup}}(\beta)]$, one has

$$g(\alpha) < \frac{\mu}{2\lambda} \qquad \text{for} \quad \alpha < \alpha_{1,\beta}$$

see (4.6), i.e.

$$\left\{\partial_\alpha\left(p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right)\right\} < 0 \tag{4.10}$$

for $\alpha < \alpha_{1,\beta}$, whereas for $\alpha > \alpha_{1,\beta}$ we obtain

$$g(\alpha) > \frac{\mu}{2\lambda} \Leftrightarrow \left\{\partial_\alpha\left(p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right)\right\} > 0 \tag{4.11}$$

see (4.6). Therefore, via (4.10) and (4.11) combined with (4.5), one has

$$\widetilde{\alpha}_\beta(\mu) = \alpha_{1,\beta} \qquad \text{for} \quad \mu \in [\mu_{1,\mathrm{inf}}(\beta), \mu_{1,\mathrm{sup}}(\beta)]. \tag{4.12}$$

Now, from (4.8) we may have two different cases (see figure 1):

(a) either there is no critical particle density, i.e. (2.20) is satisfied;
(b) or there is a *saturation* of the infinite-volume particle density $\rho^X(\beta,\alpha)$ (2.18), i.e. one has a critical particle density $\rho^X(\beta,\alpha_{\mathrm{sup}})$ (2.21).

If there is a critical particle density $\rho^X(\beta,\alpha_{\mathrm{sup}})$, then we define by

$$\mu_c(\beta) \equiv 2\lambda\rho^X(\beta,\alpha_{\mathrm{sup}}) + \alpha_{\mathrm{sup}} = 2\lambda \lim_{\alpha\to\alpha^-_{\mathrm{sup}}} \partial_\alpha p^X(\beta,\alpha) + \alpha_{\mathrm{sup}} = 2\lambda \lim_{\alpha\to\alpha^-_{\mathrm{sup}}} g(\alpha) \tag{4.13}$$

the corresponding critical chemical potential (cf (2.19) and (4.6)). Thus, for $\mu = \mu_c(\beta)$, the equality (4.9) is verified and $\widetilde{\alpha}_\beta(\mu_c(\beta)) = \alpha_{\mathrm{sup}}$. Moreover, by the convexity of $p^X(\beta,\alpha)$ as a function of $\alpha < \alpha_{\mathrm{sup}}$, one has

$$g(\alpha) < \frac{\mu}{2\lambda} \Leftrightarrow \partial_\alpha\left(p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right) < 0 \tag{4.14}$$

for any $\alpha < \alpha_{\mathrm{sup}}$ and $\mu > \mu_c(\beta)$ (4.13), see also (4.6). Consequently, assuming (2.21), the unique solution $\widetilde{\alpha}_\beta(\mu)$ of (4.5) equals

$$\widetilde{\alpha}_\beta(\mu \geqslant \mu_c(\beta)) = \alpha_{\mathrm{sup}}. \tag{4.15}$$

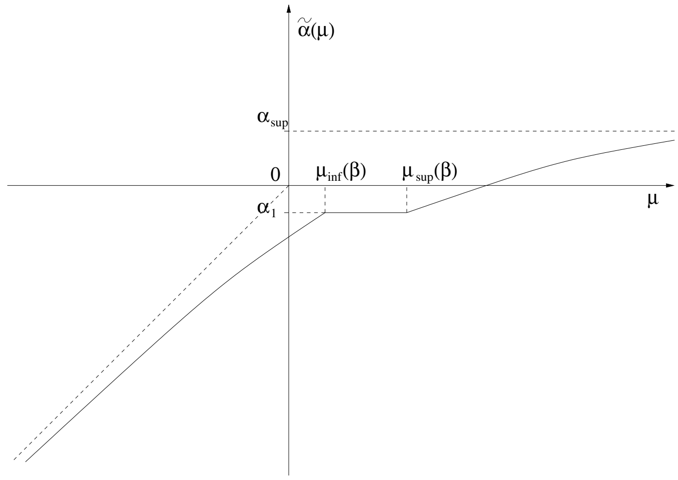

Figure 2. Illustration of the function $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$ defined by equation (4.5) with no saturation of the particle density $\rho^X(\beta, \alpha)$ (2.18), see (2.20).

To summarize (4.5)–(4.15), $\widetilde{\alpha}_\beta(\mu)$ is *unique* and *continuous* and:

1°. If there is *no saturation* of the particle density $\rho^X(\beta, \alpha)$ (2.18), i.e. (2.20) is satisfied, then

$$\begin{aligned} &(a) \text{ for } \mu \in (-\infty, \mu_{1,\inf}(\beta)], \widetilde{\alpha}_\beta(\mu) \in (-\infty, \alpha_{1,\beta}] \text{ is } \textit{stricly increasing};\\ &(b) \text{ for } \mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)], \widetilde{\alpha}_\beta(\mu) = \alpha_{1,\beta} \text{ is } \textit{constant};\\ &(c) \text{ for } \mu \geqslant \mu_{1,\sup}(\beta) \text{ (4.7)}, \widetilde{\alpha}_\beta(\mu) \in [\alpha_{1,\beta}, \alpha_{\sup}) \text{ is } \textit{stricly increasing}, \end{aligned} \tag{4.16}$$

cf figure 2.

2°. If there is a *saturation* (2.21) of the particle density $\rho^X(\beta, \alpha)$ (2.18) then

$$\begin{aligned} &(a) \text{ for } \mu \in (-\infty, \mu_{1,\inf}(\beta)], \widetilde{\alpha}_\beta(\mu) \in (-\infty, \alpha_{1,\beta}] \text{ is } \textit{stricly increasing};\\ &(b) \text{ for } \mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)], \widetilde{\alpha}_\beta(\mu) = \alpha_{1,\beta} \text{ is } \textit{constant};\\ &(c) \text{ for } \mu \in [\mu_{1,\sup}(\beta), \mu_c(\beta)], \widetilde{\alpha}_\beta(\mu) \in [\alpha_{1,\beta}, \alpha_{\sup}] \text{ is } \textit{stricly increasing};\\ &(d) \text{ for } \mu \geqslant \mu_c(\beta) \text{ (4.13)}, \widetilde{\alpha}_\beta(\mu) = \alpha_{\sup} \text{ is } \textit{constant}, \end{aligned} \tag{4.17}$$

cf figure 3.

4.2. The grand-canonical particle density

Now, we are in a position to analyse the grand-canonical particle density $\rho_\Lambda^{SX}(\beta, \mu)$ (2.9) in the thermodynamic limit.

Theorem 4.1. *Assuming condition 2.1 and (iv), we have*

$$\rho^{SX}(\beta, \mu) \equiv \lim_\Lambda \rho_\Lambda^{SX}(\beta, \mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} \tag{4.18}$$

for $(\beta, \mu) \in Q^S$ *(3.3), with* $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$ *defined by (4.5), see also (4.16) and figure 2 or (4.17) and figure 3.*

Note that (4.18) remains true even if the non-superstable Hamiltonian H_Λ^X *(2.1) verifies only (i)–(iii) of condition 2.1.*

Proof. A part of this proof is already included in the rigorous proof of theorem 3.2 (mostly given in appendix A). However, in order to clarify the arguments concerning only the particle density $\rho^{SX}(\beta, \mu)$ (4.18) in appendix A and also to complete them, we rewrite everything.

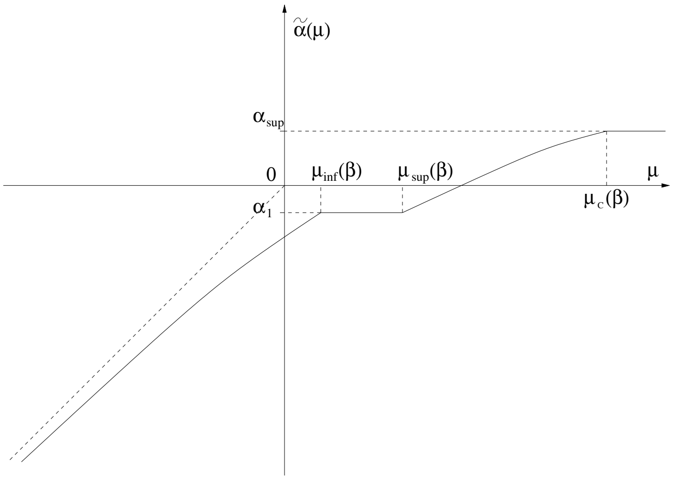

Figure 3. Illustration of the function $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$ defined by equation (4.5) with saturation of the particle density $\rho^X(\beta,\alpha)$ (2.18), see (2.21).

1°. Since the set $\left\{p_\Lambda^{SX}(\beta,\mu)\right\}_\Lambda$ (2.8) is a set of convex functions for $\mu \in \mathbb{R}$, using the Griffiths lemma [28, 29], (2.9) and (3.4), i.e. (4.5), imply, in the thermodynamic limit,

$$\rho^{SX}(\beta,\mu) = \partial_\mu p^{SX}(\beta,\mu) = \partial_\alpha \left\{ p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right\}\Bigg|_{\alpha=\widetilde{\alpha}_\beta(\mu)} \partial_\mu \widetilde{\alpha}_\beta(\mu)$$
$$+ \partial_\mu \left\{ p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right\}\Bigg|_{\alpha=\widetilde{\alpha}_\beta(\mu)}. \tag{4.19}$$

2°. Let us consider

$$\mu \in (-\infty, 2\lambda \lim_{\alpha\to\alpha_{\sup}^-} \partial_\alpha p^X(\beta,\alpha) + \alpha_{\sup}) \backslash [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$$

cf (4.7). Then, equations (4.9) and (4.19) imply

$$\rho^{SX}(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} \qquad \widetilde{\alpha}_\beta(\mu) < \alpha_{\sup} \quad \widetilde{\alpha}_\beta(\mu) \neq \alpha_{1,\beta} \tag{4.20}$$

for $\mu < \mu_{1,\inf}(\beta)$ or

$$\mu_{1,\sup}(\beta) < \mu < 2\lambda \lim_{\alpha\to\alpha_{\sup}^-} \partial_\alpha p^X(\beta,\alpha) + \alpha_{\sup}.$$

3°. Let us consider

$$\mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)].$$

Then, via (4.12) one has $\widetilde{\alpha}_\beta(\mu) = \alpha_{1,\beta}$ for $\mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$, i.e.

$$\partial_\mu \widetilde{\alpha}_\beta(\mu) = 0 \qquad \text{for} \quad \mu \in (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)). \tag{4.21}$$

By (4.20), note that

$$\begin{aligned} \lim_{\mu\to\mu_{1,\inf}^-(\beta)} \rho^{SX}(\beta,\mu) &= \frac{(\mu_{1,\inf}(\beta) - \alpha_{1,\beta})}{2\lambda} \\ \lim_{\mu\to\mu_{1,\sup}^+(\beta)} \rho^{SX}(\beta,\mu) &= \frac{(\mu_{1,\sup}(\beta) - \alpha_{1,\beta})}{2\lambda}. \end{aligned} \tag{4.22}$$

Therefore, equation (4.19) combined with (4.21) gives

$$\rho^{SX}(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda}$$

for $\mu \in (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$, which, from (4.12) and (4.22), implies

$$\rho^{SX}(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} = \frac{(\mu - \alpha_{1,\beta})}{2\lambda} \qquad \text{for} \quad \mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]. \tag{4.23}$$

4°. Assuming (2.21), let us consider $\mu \geqslant \mu_c(\beta)$ (4.13). Then, from (4.15), one has $\widetilde{\alpha}_\beta(\mu) = \alpha_{\sup}$ for $\mu \geqslant \mu_c(\beta)$, i.e.

$$\partial_\mu \widetilde{\alpha}_\beta(\mu) = 0 \qquad \text{for} \quad \mu > \mu_c(\beta). \tag{4.24}$$

Note that (4.20) gives the following limit:

$$\lim_{\mu \to \mu_c^-(\beta)} \rho^{SX}(\beta,\mu) = \frac{(\mu_c(\beta) - \alpha_{\sup})}{2\lambda}. \tag{4.25}$$

Then, from (4.19) and (4.24), we obtain

$$\rho^{SX}(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda}$$

for $\mu > \mu_c(\beta)$, which, by (4.15) and (4.25), also gives

$$\rho^{SX}(\beta,\mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} = \frac{(\mu - \alpha_{\sup})}{2\lambda} \qquad \text{for} \quad \mu \geqslant \mu_c(\beta). \tag{4.26}$$

5°. Therefore, (4.18) is a direct consequence of (4.20), (4.23) and (4.26). □

Remark 4.2. From theorem 4.1, we have

$$\rho^{SX}(\beta,\mu) : \mu \in \mathbb{R} \to (0, +\infty)$$

see also lemma A.1 in appendix A. In fact, even if $\rho^X(\beta,\alpha)$ is discontinuous for $\alpha = \alpha_{1,\beta}$, $\rho^{SX}(\beta,\mu)$ is a continuous and strictly increasing function from $\mu \in \mathbb{R}$ to $(0, +\infty)$.

Remark 4.3. By (4.5) combined with theorem 4.1, we have

$$p^{SX}(\beta,\mu) = p^X(\beta, \widetilde{\alpha}_\beta(\mu)) + \lambda[\rho^{SX}(\beta,\mu)]^2.$$

Then, according to the case verified by the model X (2.1), either (2.20) or (2.21) (cf also (4.16) and (4.17) and figures 2 and 3), one has two different corollaries from theorem 4.1.

Corollary 4.4. *If (2.20) is verified, then for $(\beta,\mu) \in Q^S$ (3.3) the unique solution $\widetilde{\alpha}_\beta(\mu) < \alpha_{\sup}$ of (4.5) verifies:*

$$\rho^{SX}(\beta,\mu) = \begin{cases} \rho^X(\beta, \widetilde{\alpha}_\beta(\mu)) & \text{for} \quad \mu \leqslant \mu_{1,\inf}(\beta) \\ \rho^X_{\inf}(\beta, \alpha_{1,\beta}) + \dfrac{\mu - \mu_{1,\inf}(\beta)}{2\lambda} & \text{for} \quad \mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)] \\ \rho^X(\beta, \widetilde{\alpha}_\beta(\mu)) & \text{for} \quad \mu \geqslant \mu_{1,\sup}(\beta) \end{cases} \tag{4.27}$$

cf figure 1(a) and figure 4, see also (4.16) and figure 2. Here $\rho^X_{\inf}(\beta, \alpha_{1,\beta})$, $\rho^X_{\sup}(\beta, \alpha_{1,\beta})$ and $\mu_{1,\inf}(\beta)$, $\mu_{1,\sup}(\beta)$ are defined by (4.2) and (4.7), respectively.

Proof. From (2.20), equation (4.9) is verified for any

$$\mu \in \mathbb{R} \backslash [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$$

cf (4.7) and (4.8). Then, from (2.19) and (4.9) we obtain

$$\left\{ \partial_\alpha \left(p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right) \right\} \bigg|_{\alpha = \widetilde{\alpha}_\beta(\mu)} = \rho^X(\beta, \widetilde{\alpha}_\beta(\mu)) - \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} = 0 \tag{4.28}$$

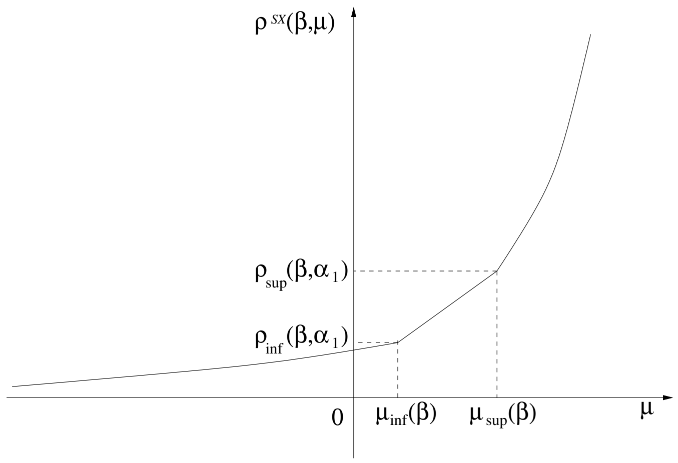

Figure 4. Illustration of the particle density $\rho^{SX}(\beta,\mu)$ (4.18) with nosaturation of the particle density $\rho^X(\beta,\alpha)$ (2.18), see (2.20) and (4.27).

for $\mu \in \mathbb{R}\backslash[\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$, which by theorem 4.1 implies (4.27).

For

$$\mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$$

$\widetilde{\alpha}_\beta(\mu) = \alpha_{1,\beta}$ (4.12) and the equality (4.27) is just a consequence of the definitions (4.2) and (4.7) combined with theorem 4.1. □

Corollary 4.5. *If there is a saturation of the particle density $\rho^X(\beta,\alpha)$ (2.18), i.e. one has a critical particle density $\rho^X(\beta,\alpha_{\sup})$ (2.21) and a corresponding critical chemical potential $\mu_c(\beta)$ (4.13), then we obtain*

$$\rho^{SX}(\beta,\mu) = \begin{cases} \rho^X(\beta,\widetilde{\alpha}_\beta(\mu)) & \text{for} \quad \mu \leqslant \mu_{1,\inf}(\beta) \\ \rho^X_{\inf}(\beta,\alpha_{1,\beta}) + \dfrac{\mu-\mu_{1,\inf}(\beta)}{2\lambda} & \text{for} \quad \mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)] \\ \rho^X(\beta,\widetilde{\alpha}_\beta(\mu)) & \text{for} \quad \mu \in [\mu_{1,\sup}(\beta), \mu_c(\beta)] \\ \rho^X(\beta,\alpha_{\sup}) + \dfrac{\mu-\mu_c(\beta)}{2\lambda} & \text{for} \quad \mu \geqslant \mu_c(\beta) \end{cases} \tag{4.29}$$

cf figures 1(b) and 5, see also (4.17) and figure 3. Here $\rho^X_{\inf}(\beta,\alpha_{1,\beta})$, $\rho^X_{\sup}(\beta,\alpha_{1,\beta})$ and $\mu_{1,\inf}(\beta)$, $\mu_{1,\sup}(\beta)$ are defined by (4.2) and (4.7), respectively.

Proof. From (2.21), equation (4.9) is verified for any

$$\mu \in (-\infty, \mu_c(\beta))\backslash[\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$$

cf (4.7) and (4.8). Hence, through (2.19) and (4.9) we obtain (4.28), which implies (4.29) for $\mu \in (-\infty, \mu_c(\beta))\backslash[\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$. From (4.2), (4.7) and (4.12) combined with theorem 4.1, we obtain (4.27) for $\mu \in [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$, whereas, for $\mu \geqslant \mu_c(\beta)$, equation (4.29) comes directly from (4.13), (4.15) and theorem 4.1. □

5. Direct applications

5.1. The fixed particle density as a parameter in the grand-canonical ensemble

Let us consider for $\rho > 0$ the *fixed* particle density in the *grand-canonical* ensemble.

On the one hand, if the grand-canonical particle density $\rho^X(\beta,\alpha)$ (2.18) is a strictly increasing function for $\alpha < \alpha_{\sup}$, then for any

$$\rho < \lim_{\alpha\to\alpha_{\sup}^-} \rho^X(\beta,\alpha) \tag{5.1}$$

there is a unique $\alpha(\rho) < \alpha_{\sup}$ such that

$$\rho^X(\beta,\alpha(\rho)) = \partial_\alpha p^X(\beta,\alpha(\rho)) = \rho \tag{5.2}$$

cf (2.18) and (2.19). If (2.21) is verified, i.e. there is a saturation of the infinite-volume particle density $\rho^X(\beta,\alpha)$ (2.18), we extend the function $\alpha(\rho)$ to $\rho \geqslant \rho^X(\beta,\alpha_{\sup})$ by

$$\alpha(\rho) \equiv \alpha_{\sup} \qquad \text{for} \quad \rho \geqslant \rho^X(\beta,\alpha_{\sup}). \tag{5.3}$$

On the other hand, from lemma A.1 in appendix A, for any $\rho > 0$ there is *always* a unique chemical potential $\mu(\rho)$ solution of the equation

$$\rho^{SX}(\beta,\mu(\rho)) = \rho \tag{5.4}$$

which also verifies

$$\widetilde{\alpha}_\beta(\mu(\rho)) = \alpha(\rho) \tag{5.5}$$

for any $\rho > 0$, see corollaries 4.4 and 4.5 combined with (5.2) and (5.3), cf also (4.16) and (4.17) and figures 2 and 3. In fact, from (5.2)–(5.5) combined with theorem 4.1, we find

$$\rho^X(\beta,\alpha(\rho)) = \rho^{SX}(\beta,\mu(\rho)) = \frac{(\mu(\rho)-\alpha(\rho))}{2\lambda} = \rho \tag{5.6}$$

which implies that

$$\mu(\rho) = 2\lambda\rho + \alpha(\rho) \tag{5.7}$$

for any $\rho > 0$, cf (5.2) and (5.3).

If we consider the thermodynamic limit of the pressures (2.8) for a fixed particle density $\rho > 0$, one obtains the following result.

Theorem 5.1. *If the non-superstable Hamiltonian H_Λ^X (2.1) verifies condition 2.1, then for any*

$$\rho < \lim_{\alpha\to\alpha_{\sup}^-} \rho^X(\beta,\alpha)$$

one has

$$p^{SX}(\beta,\mu(\rho)) = p^X(\beta,\widetilde{\alpha}_\beta(\mu(\rho))) + \lambda\rho^2 = p^X(\beta,\alpha(\rho)) + \lambda\rho^2. \tag{5.8}$$

If (2.21) is verified, then for $\rho \geqslant \rho^X(\beta,\alpha_{\sup})$,

$$p^{SX}(\beta,\mu(\rho)) = p^X(\beta,\alpha_{\sup}) + \lambda\rho^2 = p^X(\beta,\alpha(\rho)) + \lambda\rho^2 \tag{5.9}$$

where $p^X(\beta,\alpha_{\sup})$ is defined by (2.17). $\alpha(\rho)$ and $\mu(\rho)$ are defined by (5.2)–(5.4), respectively.

Proof. From (5.2)–(5.5) combined with remark 4.2 one obtains (5.8)–(5.9). □

Therefore, from theorem 5.1, for a fixed particle density $\rho > 0$, we find in the thermodynamic limit the corresponding grand-canonical pressure for the superstable model SX (2.4). Actually, this theorem has to be related to theorem 3.1:

$$f^{SX}(\beta,\rho) = f^X(\beta,\rho) + \lambda\rho^2 \qquad p^{SX}(\beta,\mu(\rho)) = p^X(\beta,\alpha(\rho)) + \lambda\rho^2 \tag{5.10}$$

for $\rho > 0$.

In spite of (5.10) note that $f^{SX}(\beta,\rho)$ and $p^{SX}(\beta,\mu(\rho))$ are of course not equal, see (5.11). We can make the same remark for $f^X(\beta,\rho)$ and $p^X(\beta,\alpha(\rho))$. However, from theorem 5.1 one can easily find again the result of theorem 3.1 assuming convexity of $f^X(\beta,\rho)$ (2.11) and $f^{SX}(\beta,\rho)$ (3.2).

Indeed, if the free-energy density $f^{SX}(\beta,\rho)$ (3.2) is a convex function for $\rho > 0$, since the pressure $p^{SX}(\beta,\mu)$ (4.5) is defined as the Legendre transformation of $f^{SX}(\beta,\rho)$, by (5.4) one also has

$$f^{SX}(\beta,\rho) = \sup_{\mu\in\mathbb{R}}\{\mu\rho - p^{SX}(\beta,\mu)\} = \mu(\rho)\rho - p^{SX}(\beta,\mu(\rho)) \tag{5.11}$$

i.e. the *weak equivalence* of canonical and grand-canonical ensembles is verified for the superstable Bose gas SX (2.4).

In the same way, using (2.15) we have

$$f^X(\beta,\rho) = \sup_{\alpha<\alpha_{\sup}}\{\alpha\rho - p^X(\beta,\alpha)\} = \alpha(\rho)\rho - p^X(\beta,\alpha(\rho)) \tag{5.12}$$

for $\rho < \lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha)$, whereas if (2.21) is satisfied then for $\rho \geqslant \rho^X(\beta,\alpha_{\sup})$

$$\partial_\alpha\{\alpha\rho - p^X(\beta,\alpha)\} = \rho - \partial_\alpha p^X(\beta,\alpha) = \rho - \rho^X(\beta,\alpha) \geqslant 0$$

and

$$f^X(\beta,\rho) = \sup_{\alpha<\alpha_{\sup}}\{\alpha\rho - p^X(\beta,\alpha)\} = \alpha_{\sup}\rho - p^X(\beta,\alpha_{\sup}) = \alpha(\rho)\rho - p^X(\beta,\alpha(\rho)) \tag{5.13}$$

for $\rho \geqslant \rho^X(\beta,\alpha_{\sup})$, see (5.3). Therefore, assuming the convexity of $f^X(\beta,\rho)$ and $f^{SX}(\beta,\rho)$ as a function of $\rho > 0$, from (5.7) and (5.11)–(5.13) the results (5.8) and (5.9) imply (3.1), i.e. theorem 3.1.

Remark 5.2. For any $\rho > 0$ the function $\rho \to \mu(\rho)$ (5.4) is always a bijective function from $\rho > 0$ to $\mu(\rho) \in \mathbb{R}$, see lemma A.1 in appendix A or (5.7). But if there is a critical density $\rho^X(\beta,\alpha_{\sup})$ (2.21) for the model X (2.1) then the function $\rho \to \alpha(\rho)$ (5.2) and (5.3) is only bijective from $\rho \leqslant \rho^X(\beta,\alpha_{\sup})$ to $\alpha(\rho) \leqslant \alpha_{\sup}$.

5.2. The PBG (1.9) and its superstabilized form, i.e. the IBG (1.13)

Hence, if we consider the superstabilization of the PBG, i.e. $H_\Lambda^X = T_\Lambda$ (1.9), then we recall that $H_\Lambda^{SX} = H_\Lambda^{\mathrm{IBG}}$ (1.13) with $\lambda = v(0)/2 > 0$. For the PBG, note, again, that $\alpha_{1,\beta}$ does not exist, i.e. $\rho^X_{\inf}(\beta,\alpha_{1,\beta}) \neq \rho^X_{\sup}(\beta,\alpha_{1,\beta})$ (4.2) and $\mu_{1,\inf}(\beta) \neq \mu_{1,\sup}(\beta)$ (4.7) do not exist.

Then the theorems 3.2 and 4.1 imply, respectively, the thermodynamic limit of the pressure

$$p_\Lambda^{\mathrm{IBG}}(\beta,\mu) = \frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta\left(H_\Lambda^{\mathrm{IBG}} - \mu N_\Lambda\right)}\right) \tag{5.14}$$

associated with the IBG (1.13), and that of the particle density $\rho_\Lambda^{\mathrm{IBG}}(\beta,\mu) \equiv \langle N_\Lambda/V\rangle_{H_\Lambda^{\mathrm{IBG}}}(\beta,\mu)$:

$$p^{\mathrm{IBG}}(\beta,\mu) \equiv \lim_\Lambda p_\Lambda^{\mathrm{IBG}}(\beta,\mu) = \inf_{\alpha<0}\left\{p^{\mathrm{PBG}}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} \tag{5.15}$$

$$\rho^{\mathrm{IBG}}(\beta,\mu) \equiv \lim_\Lambda \rho_\Lambda^{\mathrm{IBG}}(\beta,\mu) = \frac{\mu - \widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)}{2\lambda} \tag{5.16}$$

where, for $\mu \in \mathbb{R}$, $\widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)$ satisfies:

$$\inf_{\alpha<0}\left\{p^{\mathrm{PBG}}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} = p^{\mathrm{PBG}}\left(\beta,\widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)\right) + \frac{\left(\mu - \widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)\right)^2}{4\lambda}. \tag{5.17}$$

Here, $\alpha_{\sup} = 0$ and

$$\begin{aligned} p^{\mathrm{PBG}}(\beta, \alpha) &= \frac{1}{\beta(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \ln(1 - \mathrm{e}^{-\beta(\varepsilon_k - \alpha)})^{-1} \\ \rho^{\mathrm{PBG}}(\beta, \alpha) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \, (\mathrm{e}^{\beta(\varepsilon_k - \alpha)} - 1)^{-1} \end{aligned} \tag{5.18}$$

are, respectively, the pressure and the particle density associated with the PBG (1.9) in the *grand-canonical* ensemble (β, α).

Since there is no critical density for the PBG in dimensions $d < 3$, i.e. (2.20) is satisfied for $d < 3$, then via corollary 4.4 (if we consider the non-existence of $\alpha_{1,\beta}$), we find that the unique solution $\widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu) < 0$ of (5.17) verifies

$$\rho^{\mathrm{IBG}}(\beta, \mu) = \rho^{\mathrm{PBG}}\left(\beta, \widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \left(\mathrm{e}^{\beta(\varepsilon_k - \widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu))} - 1\right)^{-1} \tag{5.19}$$

for $(\beta, \mu) \in Q^S$ (3.3).

Nevertheless, for the PBG with $d \geqslant 3$, there is a critical density

$$\rho^{\mathrm{PBG}}(\beta, 0) \equiv \sup_{\alpha < 0} \rho^{\mathrm{PBG}}(\beta, \alpha) = \lim_{\alpha \to 0^-} \rho^{\mathrm{PBG}}(\beta, \alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \, (\mathrm{e}^{\beta\varepsilon_k} - 1)^{-1} < +\infty \tag{5.20}$$

i.e. (2.21) is verified ($d \geqslant 3$). Then, by corollary 4.5 (if we consider the non-existence of $\alpha_{1,\beta}$) one obtains

$$\rho^{\mathrm{IBG}}(\beta, \mu) = \begin{cases} \rho^{\mathrm{PBG}}(\beta, \widetilde{\alpha}_\beta^{\mathrm{PBG}}(\mu)) & \text{for} \quad \mu \leqslant \mu_c^{\mathrm{IBG}}(\beta) \equiv 2\lambda \rho^{\mathrm{PBG}}(\beta, 0) \\ \rho^{\mathrm{PBG}}(\beta, 0) + \dfrac{\mu - \mu_c^{\mathrm{IBG}}(\beta)}{2\lambda} = \dfrac{\mu}{2\lambda} & \text{for} \quad \mu \geqslant \mu_c^{\mathrm{IBG}}(\beta). \end{cases} \tag{5.21}$$

Note that (5.15)–(5.21) are just illustrations of the general results of this paper. The thermodynamic behaviour of the IBG (1.13) was already detailed in [15–20].

For a fixed particle density $\rho > 0$, the theorem 5.1 also gives

$$p^{\mathrm{IBG}}(\beta, \mu^{\mathrm{IBG}}(\rho)) = p^{\mathrm{PBG}}(\beta, \alpha^{\mathrm{PBG}}(\rho)) + \lambda \rho^2 \qquad \rho < \lim_{\alpha \to 0^-} \rho^{\mathrm{PBG}}(\beta, \alpha) \tag{5.22}$$

see (5.18) and if $d \geqslant 3$ then for $\rho \geqslant \rho^{\mathrm{PBG}}(\beta, 0)$ (5.20),

$$p^{\mathrm{IBG}}(\beta, \mu^{\mathrm{IBG}}(\rho)) = p^{\mathrm{PBG}}(\beta, 0) + \lambda \rho^2. \tag{5.23}$$

Here $\alpha^{\mathrm{PBG}}(\rho)$ and $\mu^{\mathrm{IBG}}(\rho)$ satisfy

$$\begin{aligned} &\rho^{\mathrm{IBG}}(\beta, \mu^{\mathrm{IBG}}(\rho)) = \rho^{\mathrm{PBG}}(\beta, \alpha^{\mathrm{PBG}}(\rho)) = \rho \qquad \rho < \lim_{\alpha \to 0^-} \rho^{\mathrm{PBG}}(\beta, \alpha) \\ &\rho^{\mathrm{IBG}}(\beta, \mu^{\mathrm{IBG}}(\rho)) = \rho \qquad \rho \geqslant \rho^{\mathrm{PBG}}(\beta, 0) \qquad d \geqslant 3 \end{aligned} \tag{5.24}$$

see (5.2) and (5.4). From (5.7), note that one has

$$\mu^{\mathrm{IBG}}(\rho) = 2\lambda\rho + \alpha^{\mathrm{PBG}}(\rho) \tag{5.25}$$

where, if $d \geqslant 3$ (cf (5.20)),

$$\alpha^{\mathrm{PBG}}(\rho \geqslant \rho^{\mathrm{PBG}}(\beta, 0)) \equiv 0 \tag{5.26}$$

see (5.3).

6. Concluding remarks

By adding the 'forward scattering' interaction (1.10) or the interaction (1.14), we propose a method (2.4) of superstabilization for the non-superstable Hamiltonians H_Λ^X (2.1) which verifies some assumptions such as condition 2.1. Then, we show that the standard canonical and grand-canonical thermodynamic functions (2.6)–(2.9) of the new model SX (2.4) depend drastically on the original Hamiltonian H_Λ^X (2.1) (see theorems 3.1, 3.2, 4.2 and also corollaries 4.4 and 4.5).

Condition 2.1 *only* represents some *sufficient* conditions. The main condition is assumption (iii) of condition 2.1, i.e. the weak equivalence of ensembles (2.15), see theorems 3.1, 3.2 and 4.1. If we assume now *only* (ii) of condition 2.1 and (iv) as defined in section 4, note that the weak equivalence of ensembles (2.15) may not be verified for $\rho \in \left[\rho_{\mathrm{inf}}^X(\beta, \alpha_{1,\beta}), \rho_{\mathrm{sup}}^X(\beta, \alpha_{1,\beta})\right]$ (4.2) or/and $\rho \geqslant \rho^X(\beta, \alpha_{\mathrm{sup}})$ (2.21), see (4.1)–(4.4). In this case, the results of theorems 3.2, 4.1 and of corollaries 4.4 and 4.5 only remain correct for

$$\begin{aligned} \mu \in I_\mu(\beta) &\equiv \{\mu : \rho^X(\beta, \mu) \in I_\rho(\beta)(4.1)\} \\ &= (0, 2\lambda \lim_{\alpha \to \alpha_{\mathrm{sup}}^-} \rho^X(\beta, \alpha) + \alpha_{\mathrm{sup}}) \backslash [\mu_{1,\mathrm{inf}}(\beta), \mu_{1,\mathrm{sup}}(\beta)] \end{aligned} \tag{6.1}$$

with $\mu_{1,\mathrm{inf}}(\beta), \mu_{1,\mathrm{sup}}(\beta)$ defined by (4.7). For $\mu \in \mathbb{R}\backslash I_\mu(\beta)$, one has to know *more precisely* the behaviour of $f^X(\beta, \rho)$ (2.11) as a function of $\rho \in \mathbb{R}_+ \backslash I_\rho(\beta)$.

Applying the superstabilization (2.4) to the PBG (1.9), one obtains the IBG (1.13). Then, if we use the results mentioned above, again we find all the thermodynamic behaviour of the IBG, cf (5.15)–(5.26), see [15–20, 26]. Considering the first (superstable) full Hamiltonian H_Λ (1.4):

$$H_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2 \in \Lambda^*, q \in \Lambda^* \backslash \{0\}} v(q) a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} + \frac{v(0)}{2V} \sum_{k_1, k_2 \in \Lambda^*} a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1}$$

also note that the general results described above imply that the thermodynamic functions of this Bose gas could be evaluated from those of the model

$$\widehat{H}_\Lambda \equiv \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2 \in \Lambda^*, q \in \Lambda^* \backslash \{0\}} v(q) a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \tag{6.2}$$

if we assume that the Hamiltonian $\widehat{H}_\Lambda$ (6.2) satisfies some sufficient conditions such as the *weak equivalence of ensemble* (2.15).

This procedure of superstabilization (2.4) ensures, in the thermodynamic limit, the *strong equivalence* of ensembles (canonical/grand-canonical): for a fixed total particle density $\rho > 0$, the infinite-volume SX Gibbs state in the grand-canonical ensemble coincides with that in the canonical ensemble, since the corresponding infinite-volume particle density $\rho^{SX}(\beta, \mu)$ (4.18) is bijective from $\mu \in \mathbb{R}$ to $(0, +\infty)$, cf remarks 4.2 and 5.2. In fact, in a subsequent paper, we will show that this is done without destroying the 'fundamental' thermodynamic properties issued from the Bose system X (2.1). In particular, the phenomenona of Bose condensation are the same in both models X (2.1) and SX (2.4) (see as an example [27, 33]). Finally, using the thermodynamic behaviour of the superstable system SX in the *grand-canonical* ensemble, we will explain that this method of superstabilization (2.4) also allows us to study the original model X in the *canonical* ensemble.

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Appendix A

From (2.2) and (2.13), the Hamiltonian H_Λ^{SX} (2.4) is superstable [1] for any box Λ, cf (2.5), which implies the existence of the (infinite-volume) pressure $p^{SX}(\beta,\mu)$ for any $\mu \in \mathbb{R}$ and $\beta > 0$. The pressure $p^{SX}(\beta,\mu)$ (3.4) is the Legendre transformation of $f^{SX}(\beta,\rho)$ (3.1) and (3.2):

$$p^{SX}(\beta,\mu) = \sup_{\rho>0}\{\mu\rho - f^{SX}(\beta,\rho)\} = \sup_{\rho>0}\{\mu\rho - \lambda\rho^2 - f^X(\beta,\rho)\}. \tag{A.1}$$

Then, via (A.9),

$$\widetilde{\rho} = \rho^{SX}(\beta,\mu) \tag{A.2}$$

is a solution of

$$p^{SX}(\beta,\mu) = \sup_{\rho>0}\{\mu\rho - f^{SX}(\beta,\rho)\} = \mu\widetilde{\rho} - f^{SX}(\beta,\widetilde{\rho}) = \mu\widetilde{\rho} - \lambda\widetilde{\rho}^2 - f^X(\beta,\widetilde{\rho}). \tag{A.3}$$

Note that, generally the solution for (A.3) would not be unique if $\rho^{SX}(\beta,\mu)$ were not continuous or if $f^{SX}(\beta,\rho)$ were not strictly convex. Straightforward calculations give

$$\inf_{\alpha<\alpha_{\sup}}\left\{\alpha\rho + \frac{(\mu-\alpha)^2}{4\lambda} - f^X(\beta,\rho)\right\} = \mu\rho - \lambda\rho^2 - f^X(\beta,\rho) \tag{A.4}$$

and, thus, (A.1) takes the form

$$p^{SX}(\beta,\mu) = \sup_{\rho>0}\left\{\inf_{\alpha<\alpha_{\sup}}\left\{\alpha\rho + \frac{(\mu-\alpha)^2}{4\lambda} - f^X(\beta,\rho)\right\}\right\}. \tag{A.5}$$

Note that, in general, $\sup_{\rho>0}$ and $\inf_{\alpha<\alpha_{\sup}}$ do not commute. Actually, (following a remark of one referee), via (A.5) one has

$$p^{SX}(\beta,\mu) \leqslant \inf_{\alpha<\alpha_{\sup}}\left\{\sup_{\rho>0}\left\{\alpha\rho + \frac{(\mu-\alpha)^2}{4\lambda} - f^X(\beta,\rho)\right\}\right\} \tag{A.6}$$

and from (2.15) we deduce

$$\begin{aligned} p^{SX}(\beta,\mu) &\leqslant \inf_{\alpha<\alpha_{\sup}}\left\{\sup_{\rho>0}\{\alpha\rho - f^X(\beta,\rho)\} + \frac{(\mu-\alpha)^2}{4\lambda}\right\} \\ &= \inf_{\alpha<\alpha_{\sup}}\left\{p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} = p^X(\beta,\widetilde{\alpha}_\beta(\mu)) + \frac{(\mu-\widetilde{\alpha}_\beta(\mu))^2}{4\lambda}. \end{aligned} \tag{A.7}$$

For a gas X that also verifies assumption (iv) as defined in section 4, a complete study of the function $\widetilde{\alpha}_\beta(\mu)$ is given by (4.5)–(4.17). Note that

$$\partial_\mu\left\{\inf_{\alpha<\alpha_{\sup}}\left\{p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\}\right\} = \frac{(\mu-\widetilde{\alpha}_\beta(\mu))}{2\lambda} \geqslant \frac{(\mu-\alpha_{\sup})}{2\lambda}. \tag{A.8}$$

For further details concerning (A.8), see the proof of theorem 4.1.

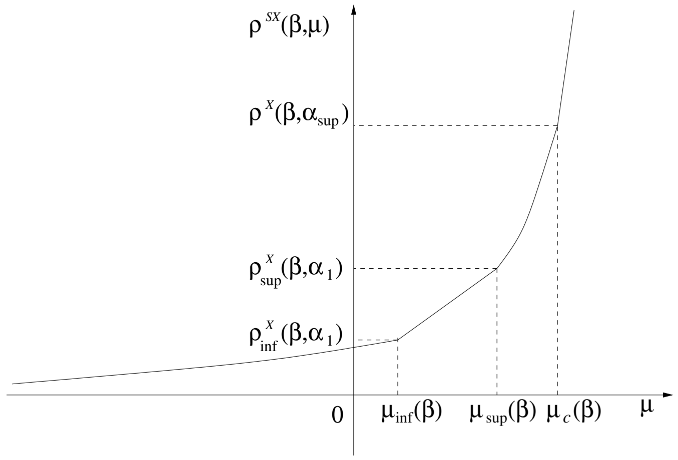

Figure 5. Illustration of the particle density $\rho^{SX}(\beta,\mu)$ (4.18) with saturation of the particle density $\rho^{X}(\beta,\alpha)$ (2.18), see (2.21) and (4.29).

Lemma A.1. *The (infinite-volume) particle density*

$$\rho^{SX}(\beta,\mu) \equiv \lim_{\Lambda} \rho^{SX}_{\Lambda}(\beta,\mu) = \lim_{\Lambda} \partial_{\mu} p^{SX}_{\Lambda}(\beta,\mu) = \partial_{\mu} p^{SX}(\beta,\mu) \tag{A.9}$$

is an increasing function for $\mu \in \mathbb{R}$, verifying

$$\lim_{\mu\to-\infty} \rho^{SX}(\beta,\mu) = 0 \qquad \lim_{\mu\to+\infty} \rho^{SX}(\beta,\mu) = +\infty \tag{A.10}$$

i.e.

$$\rho^{SX}(\beta,\mu) : \mu \in \mathbb{R} \to (0,+\infty). \tag{A.11}$$

Proof. The pressure $p^{SX}(\beta,\mu) \geqslant 0$ is an increasing convex function for $\mu \in \mathbb{R}$ and

$$\lim_{\mu\to-\infty} p^{SX}(\beta,\mu) = a \geqslant 0 \qquad \lim_{\mu\to+\infty} p^{SX}(\beta,\mu) = +\infty. \tag{A.12}$$

Through the Griffiths lemma [28, 29] applied to the sequence $\left\{p^{SX}_{\Lambda}(\beta,\mu)\right\}_{\Lambda}$ (2.8) of convex functions for $\mu \in \mathbb{R}$, by (2.9) one obtains (A.9). From (A.2) and (A.3), $\rho = \rho^{SX}(\beta,\mu)$ is solution of the equation

$$\mu = 2\lambda\rho + \partial_{\rho} f^{X}(\beta,\rho). \tag{A.13}$$

Since the function $f^{X}(\beta,\rho)$ (2.11) is convex for $\rho > 0$, then for $\mu \to +\infty$, the solution $\rho^{SX}(\beta,\mu)$ of (A.13) diverges. Consequently, considering also (A.12), $\rho^{SX}(\beta,\mu)$ is an increasing function for $\mu \in \mathbb{R}$ satisfying (A.10) and (A.11), see figure 5. □

Lemma A.2. *Let us consider $0 \leqslant \rho_1 < \rho_2 \leqslant +\infty$. If the (infinite-volume) free-energy density $f^{X}(\beta,\rho)$ (2.11) is strictly convex for $\rho \in (\rho_1,\rho_2) \subset [0,+\infty)$ and if $\partial_{\rho} f^{X}(\beta,\rho)$ is a continuous function for $\rho \in (\rho_1,\rho_2)$, then*

$$p^{SX}(\beta,\mu) \equiv \lim_{\Lambda} p^{SX}_{\Lambda}(\beta,\mu) = \inf_{\alpha<\alpha_{\sup}} \left\{ p^{X}(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right\} \tag{A.14}$$

for any $\mu \in (\mu_1,\mu_2)$ with $-\infty \leqslant \mu_1 < \mu_2 \leqslant +\infty$ defined by

$$\lim_{\mu\to\mu_1^+} \rho^{SX}(\beta,\mu) = \rho_1 \qquad \text{and} \qquad \lim_{\mu\to\mu_2^-} \rho^{SX}(\beta,\mu) = \rho_2 \tag{A.15}$$

see lemma A.1.

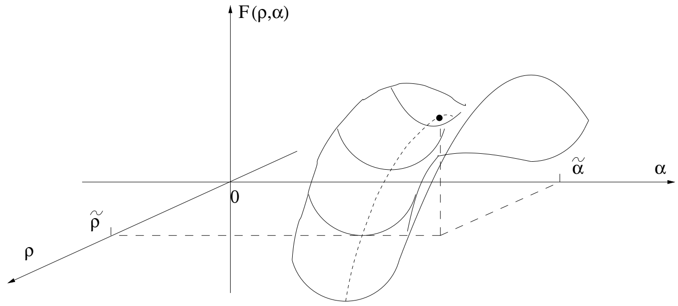

Figure 6. Illustration of the function $F(\rho, \alpha)$.

Proof. We consider $\mu \in (\mu_1, \mu_2)$, i.e. $\widetilde{\rho} = \rho^{SX}(\beta, \mu) \in (\rho_1, \rho_2)$, see lemma A.1 and (A.15). If $\rho \in (\rho_1, \rho_2)$, then $f^X(\beta, \rho)$ (2.11) is strictly convex and so the function $F(\rho, \alpha)$ defined by

$$F(\rho, \alpha) \equiv \alpha\rho + \frac{(\mu - \alpha)^2}{4\lambda} - f^X(\beta, \rho) \qquad \alpha < \alpha_{\mathrm{sup}} \quad \rho > 0 \tag{A.16}$$

is a strictly concave function for $\rho \in (\rho_1, \rho_2)$ and a strictly convex function for $\alpha < \alpha_{\mathrm{sup}}$. Then, we obtain the *uniqueness* of the stationary point $(\widetilde{\rho}, \widetilde{\alpha})$ corresponding to

$$\partial_\alpha F(\widetilde{\rho}, \widetilde{\alpha}) = \widetilde{\rho} - \frac{(\mu - \widetilde{\alpha})}{2\lambda} = 0 \qquad \partial_\rho F(\widetilde{\rho}, \widetilde{\alpha}) = 0 \tag{A.17}$$

for $\rho \in (\rho_1, \rho_2)$ and $\alpha < \alpha_{\mathrm{sup}}$, see figure 6.

Moreover, from (A.1)–(A.5) and (A.16), for $\mu \in (\mu_1, \mu_2)$ we have

$$p^{SX}(\beta, \mu) = \sup_{\rho>0} \left\{ \inf_{\alpha<\alpha_{\mathrm{sup}}} \{F(\rho, \alpha)\} \right\} = \sup_{\rho\in(\rho_1,\rho_2)} \left\{ \inf_{\alpha<\alpha_{\mathrm{sup}}} \{F(\rho, \alpha)\} \right\}. \tag{A.18}$$

Therefore, by (A.17) for $\rho \in (\rho_1, \rho_2)$ and $\alpha < \alpha_{\mathrm{sup}}$, equation (A.18) implies

$$p^{SX}(\beta, \mu) = \sup_{\rho\in(\rho_1,\rho_2)} \left\{ \inf_{\alpha<\alpha_{\mathrm{sup}}} \{F(\rho, \alpha)\} \right\} = F(\widetilde{\rho}, \widetilde{\alpha}) = \inf_{\alpha<\alpha_{\mathrm{sup}}} \left\{ \sup_{\rho\in(\rho_1,\rho_2)} \{F(\rho, \alpha)\} \right\} \tag{A.19}$$

for $\mu \in (\mu_1, \mu_2)$. Since $f^X(\beta, \rho)$ is strictly convex for $\rho \in (\rho_1, \rho_2)$, one obtains

$$-\infty \leqslant \alpha_1 \equiv \lim_{\rho\to\rho_1^+} \partial_\rho f^X(\beta, \rho) < \partial_\rho f^X(\beta, \rho) < \alpha_2 \equiv \lim_{\rho\to\rho_2^-} \partial_\rho f^X(\beta, \rho) \leqslant \alpha_{\mathrm{sup}} \tag{A.20}$$

for any $\rho \in (\rho_1, \rho_2)$. Then, via (2.15) and (2.19), note that

$$\widehat{\rho} = \rho^X(\beta, \alpha) = \partial_\alpha p^X(\beta, \alpha) \tag{A.21}$$

is the only solution for $\alpha \in (\alpha_1, \alpha_2)$ of the equation

$$p^X(\beta, \alpha) = \sup_{\rho>0} \{\alpha\rho - f^X(\beta, \rho)\} = \sup_{\rho\in(\rho_1,\rho_2)} \{\alpha\rho - f^X(\beta, \rho)\} = \alpha\widehat{\rho} - f^X(\beta, \widehat{\rho}) \tag{A.22}$$

i.e.

$$\alpha = \partial_\rho f^X(\beta, \widehat{\rho}) \in (\alpha_1, \alpha_2). \tag{A.23}$$

In fact, if $\alpha \leqslant \alpha_1$, then

$$\sup_{\rho\in(\rho_1,\rho_2)} \{\alpha\rho - f^X(\beta, \rho)\} = \lim_{\rho\to\rho_1^+} \{\alpha\rho - f^X(\beta, \rho)\} \tag{A.24}$$

whereas, if $\alpha_2 \leqslant \alpha \leqslant \alpha_{\mathrm{sup}}$,

$$\sup_{\rho\in(\rho_1,\rho_2)} \{\alpha\rho - f^X(\beta, \rho)\} = \lim_{\rho\to\rho_2^-} \{\alpha\rho - f^X(\beta, \rho)\}. \tag{A.25}$$

Therefore, if we consider the only stationary point $(\widetilde{\rho}, \widetilde{\alpha})$ (A.17) for $\mu \in (\mu_1, \mu_2)$, by (A.16) and (A.19), equations (A.2) and (A.3) and (A.24) and (A.25) imply that

$$\text{if} \quad \widetilde{\alpha} \leqslant \alpha_1 \quad \text{then} \quad \widetilde{\rho} = \lim_{\mu \to \mu_1^+} \rho^{SX}(\beta, \mu) = \rho_1 \qquad \text{and}$$
$$\text{if} \quad \widetilde{\alpha} \geqslant \alpha_2 \quad \text{then} \quad \widetilde{\rho} = \lim_{\mu \to \mu_2^-} \rho^{SX}(\beta, \mu) = \rho_2$$

which contradicts $\mu \in (\mu_1, \mu_2)$, i.e. $\widetilde{\rho} = \rho^{SX}(\beta, \mu) \in (\rho_1, \rho_2)$, see lemma A.1 and (A.15). Hence, for $\mu \in (\mu_1, \mu_2)$, $\widetilde{\alpha} \in (\alpha_1, \alpha_2)$ and by (A.19) we find

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\text{sup}}} \left\{ \sup_{\rho \in (\rho_1, \rho_2)} \{F(\rho, \alpha)\} \right\} = \inf_{\alpha \in (\alpha_1, \alpha_2)} \left\{ \sup_{\rho \in (\rho_1, \rho_2)} \{F(\rho, \alpha)\} \right\} = F(\widetilde{\rho}, \widetilde{\alpha}) \tag{A.26}$$

for $\mu \in (\mu_1, \mu_2)$. By (A.16) and (A.22) we have

$$\sup_{\rho \in (\rho_1, \rho_2)} \{F(\rho, \alpha)\} = \sup_{\rho \in (\rho_1, \rho_2)} \{\alpha\rho - f^X(\beta, \rho)\} + \frac{(\mu - \alpha)^2}{4\lambda} = p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda}$$

for $\alpha \in (\alpha_1, \alpha_2)$, which by (A.26) implies

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\text{sup}}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = p^X(\beta, \widetilde{\alpha}_\beta(\mu)) + \frac{(\mu - \widetilde{\alpha}_\beta(\mu))^2}{4\lambda} \tag{A.27}$$

for $\mu \in (\mu_1, \mu_2)$. □

Remark A.3. From (A.9), (A.2), (A.21) and (A.22) combined with (A.26), and via (A.27), note that the only stationary point $(\widetilde{\rho}, \widetilde{\alpha})$ (A.17) verifies $\widetilde{\alpha} = \widetilde{\alpha}_\beta(\mu) \in (\alpha_1, \alpha_2)$ and

$$\widetilde{\rho} = \rho^{SX}(\beta, \mu) = \rho^X(\beta, \widetilde{\alpha}_\beta(\mu)) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} = \partial_\mu p^{SX}(\beta, \mu) \tag{A.28}$$

for $\mu \in (\mu_1, \mu_2)$ (A.15).

From (2.15) in condition 2.1, note that the (infinite-volume) free-energy density $f^X(\beta, \rho)$ (2.11), as a function of $\rho > 0$, is convex but not necessarily strictly so.

Lemma A.4. *Let us consider* $0 < \rho_1 < \rho_2 \leqslant +\infty$. *If the free-energy density* $f^X(\beta, \rho)$ *(2.11) is not strictly convex for* $\rho \in [\rho_1, \rho_2) \subset (0, +\infty)$, *i.e.* $f^X(\beta, \rho)$ *is a straight line for* $\in [\rho_1, \rho_2)$, *then*

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\text{sup}}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{A.29}$$

for any $\mu \in [\mu_1, \mu_2)$ *with* $-\infty < \mu_1 < \mu_2 \leqslant +\infty$ *defined by (A.15).*

Proof. Let us consider $\mu \in [\mu_1, \mu_2)$, i.e. $\widetilde{\rho} = \rho^{SX}(\beta, \mu) \in [\rho_1, \rho_2 \leqslant +\infty)$, see lemma A.1 and (A.15). From (A.1)–(A.3), for $\mu \in [\mu_1, \mu_2)$ we have

$$p^{SX}(\beta, \mu) = \sup_{\rho > 0} \{\mu\rho - f^{SX}(\beta, \rho)\} = \sup_{\rho \in [\rho_1, \rho_2)} \{\mu\rho - \lambda\rho^2 - f^X(\beta, \rho)\}. \tag{A.30}$$

Through (2.15), the free-energy density $f^X(\beta, \rho)$ is convex for $\rho > 0$, but not strictly for $\rho \in [\rho_1, \rho_2)$. Then,

$$\forall \rho \in [\rho_1, \rho_2) \qquad \left\{ \begin{array}{l} \partial_\rho f^X(\beta, \rho) = \partial_\rho f^X(\beta, \rho_1) = \alpha_1 \leqslant \alpha_{\text{sup}} \\ f^X(\beta, \rho) = \alpha_1(\rho - \rho_1) + f^X(\beta, \rho_1) \end{array} \right\} \tag{A.31}$$

and from (A.22),

$$p^X(\beta, \alpha_1) = \alpha_1\rho_1 - f^X(\beta, \rho_1) = \alpha_1\widehat{\rho} - f^X(\beta, \widehat{\rho}) \tag{A.32}$$

for any $\widehat{\rho} \in [\rho_1, \rho_2)$. Actually, the particle density $\rho^X(\beta, \alpha)(\rho^X(\beta, \alpha_1) \in [\rho_1, \rho_2))$ (2.18) and (2.19) is not continuous for $\alpha = \alpha_1$. Consequently, via (A.30)–(A.32), we obtain

$$p^{SX}(\beta, \mu) = p^X(\beta, \alpha_1) + \sup_{\rho \in [\rho_1, \rho_2)} \{(\mu - \alpha_1)\rho - \lambda\rho^2\} = p^X(\beta, \alpha_1) + \frac{(\mu - \alpha_1)^2}{4\lambda} \tag{A.33}$$

for $\mu \in [\mu_1, \mu_2)$, which from (A.9) implies

$$\rho^{SX}(\beta, \mu) = \frac{(\mu - \alpha_1)}{2\lambda} \in (\rho_1, \rho_2) \tag{A.34}$$

for $\mu \in (\mu_1, \mu_2)$, i.e.

$$\mu_1 = 2\lambda\rho_1 + \alpha_1 \qquad \mu_2 = 2\lambda\rho_2 + \alpha_1 \leqslant +\infty. \tag{A.35}$$

Since $\rho^X(\beta, \alpha)$ is an increasing function of $\alpha < \alpha_{\sup}$, from (A.35) we deduce that

$$\forall \alpha \leqslant \alpha_1 \quad \partial_\alpha \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \leqslant 0$$

$$\forall \alpha \geqslant \alpha_1 \quad \partial_\alpha \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \geqslant 0$$

for $\mu \in [\mu_1, \mu_2)$, i.e. one has

$$p^{SX}(\beta, \mu) = p^X(\beta, \alpha_1) + \frac{(\mu - \alpha_1)^2}{4\lambda} = \inf_{\alpha < \alpha_{\sup}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{A.36}$$

for $\mu \in [\mu_1, \mu_2)$, cf (A.33). □

Lemma A.5. *If $\rho_1 \in (0, +\infty)$ is such that $\partial_\rho f^X(\beta, \rho)$ is not continuous for $\rho = \rho_1$, then*

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\sup}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{A.37}$$

for

$$\mu \in \{\mu \in \mathbb{R} : \rho^{SX}(\beta, \mu) = \rho_1\} = [\mu_1, \widetilde{\mu}_1] \tag{A.38}$$

see lemma A.1.

Proof. Let us consider

$$\alpha_1 \equiv \lim_{\rho \to \rho_1^-} \partial_\rho f^X(\beta, \rho) < \widetilde{\alpha}_1 \equiv \lim_{\rho \to \rho_1^+} \partial_\rho f^X(\beta, \rho).$$

Then, from (2.15) we have

$$\begin{aligned} &[\alpha_1, \widetilde{\alpha}_1] = \{\alpha \in \mathbb{R} : \rho^X(\beta, \alpha) = \rho_1\} \\ &p^X(\beta, \alpha) = \sup_{\rho > 0} \{\alpha\rho - f^X(\beta, \rho)\} = \alpha\rho_1 - f^X(\beta, \rho_1) \qquad \text{for} \quad \alpha \in [\alpha_1, \widetilde{\alpha}_1] \end{aligned} \tag{A.39}$$

i.e. $p^X(\beta, \alpha)$ is not strictly convex for $\alpha \in [\alpha_1, \widetilde{\alpha}_1]$. If we use (A.5), we obtain

$$p^{SX}(\beta, \mu) = \sup_{\rho > 0} \left\{ \inf_{\alpha \in [\alpha_1(\rho), \widetilde{\alpha}_1(\rho)]} \left\{ \alpha\rho - f^X(\beta, \rho) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \right\} \tag{A.40}$$

for $\mu \in [\mu_1, \widetilde{\mu}_1]$ with

$$\begin{aligned} &\mu_1 \equiv 2\lambda\rho_1 + \alpha_1 \qquad \widetilde{\mu}_1 \equiv 2\lambda\rho_1 + \widetilde{\alpha}_1 \\ &\alpha_1(\rho) \equiv 2\lambda(\rho_1 - \rho) + \alpha_1 \qquad \widetilde{\alpha}_1(\rho) \equiv 2\lambda(\rho_1 - \rho) + \widetilde{\alpha}_1. \end{aligned} \tag{A.41}$$

Since $f^X(\beta, \rho)$ is convex for $\rho > 0$ (cf (2.15)), $\forall \alpha \in [\alpha_1(\rho), \widetilde{\alpha}_1(\rho)]$,

$$\partial_\rho \left\{ \alpha\rho - f^X(\beta, \rho) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} > 0 \tag{A.42}$$

for $0 < \rho < \rho_1$, whereas for $\rho > \rho_1$

$$\partial_\rho \left\{ \alpha\rho - f^X(\beta, \rho) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} < 0. \tag{A.43}$$

Consequently, from (A.40) and (A.41) combined with (A.42) and (A.43), we obtain

$$\begin{aligned} p^{SX}(\beta, \mu) &= \inf_{\alpha \in [\alpha_1(\rho_1), \widetilde{\alpha}_1(\rho_1)]} \left\{ \alpha\rho_1 - f^X(\beta, \rho_1) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \\ &= \inf_{\alpha \in [\alpha_1, \widetilde{\alpha}_1]} \left\{ \alpha\rho_1 - f^X(\beta, \rho_1) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \end{aligned} \tag{A.44}$$

for $\mu \in [\mu_1, \widetilde{\mu}_1]$. Since

$$\begin{aligned} \inf_{\alpha \in [\alpha_1, \widetilde{\alpha}_1]} \left\{ \alpha\rho_1 - f^X(\beta, \rho_1) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} &= \inf_{\alpha < \alpha_{\sup}} \left\{ \alpha\rho_1 - f^X(\beta, \rho_1) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \\ &= \mu\rho_1 - \lambda\rho_1^2 - f^X(\beta, \rho_1) \end{aligned}$$

for $\mu \in [\mu_1, \widetilde{\mu}_1]$, from (A.39) and (A.44), we deduce

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\sup}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = \mu\rho_1 - \lambda\rho_1^2 - f^X(\beta, \rho_1)$$

for $\mu \in [\mu_1, \widetilde{\mu}_1]$, i.e. for $\mu \in \{\mu \in \mathbb{R} : \rho^{SX}(\beta, \mu) = \rho_1\}$, cf (A.2) and (A.3). □

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